Rotating waves in rings of coupled oscillators

Carlo R. Laing,

Department of Applied Mathematics and Theoretical Physics,

University of Cambridge, Silver Street, Cambridge CB3 9EW, U.K.

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Current address: Department of Mathematics and Statistics,

University of Surrey, Guildford GU2 5XH, United Kingdom.

Telephone: (01483) 300800 extn 2620

Fax: (01483) 259385

Email: `mas1cl@ee.surrey.ac.uk`

Abstract

In this paper we discuss the types of stable oscillation created via Hopf bifurcations for a ring of identical nonlinear oscillators, each of which is diffusively and symmetrically coupled to both its neighbours, and which, when uncoupled, undergo a supercritical Hopf bifurcation creating a stable periodic orbit as a parameter, λ, is increased.

We show that for small enough coupling, the only stable rotating waves produced are either one or a conjugate pair, depending on the parity of the number of oscillators in the ring and the sign of the coupling constant, and that the magnitude of the phase difference between neighbouring oscillators for these rotating waves is either zero (i.e. the oscillators are synchronised) or the maximum possible, depending on the sign of the coupling constant. These branches of rotating waves are produced supercritically.

1 Introduction

determine the types of oscillation produced. This was the approach used by Collins and Stewart [3] in relation to the modelling of animal gaits. These authors used results from

2 Presentation of system

We assume that our system of N coupled oscillators in a ring is governed by equations of the form

$$\dot{z}_j \equiv \frac{dz_j}{dt} = (\lambda + i\Omega)z_j + F_2(z_j, \bar{z}_j) + F_3(z_j, \bar{z}_j) + (\epsilon_r + i\epsilon_i)(2z_j - z_{j+1} - z_{j-1}) \tag{1}$$

where $z_j \in \mathrm{C}$; $\lambda, \Omega, \epsilon_r, \epsilon_i \in \mathrm{R}$, $i^2 = -1$, the subscripts (which label the oscillators) are taken mod N, and F_2 and F_3 contain the second and third order terms, respectively, in the Taylor series expansion of the vector field of an uncoupled oscillator. To be more explicit, we write

$$F_2(z_j, \bar{z}_j) = \alpha_1 z_j^2 + \alpha_2 z_j \bar{z}_j + \alpha_3 \bar{z}_j^2$$

and

$$F_3(z_j, \bar{z}_j) = \beta_1 z_j^3 + \beta_2 z_j^2 \bar{z}_j + \beta_3 z_j \bar{z}_j^2 + \beta_4 \bar{z}_j^3$$

where the α's and β's are complex constants. We take λ as our bifurcation parameter, which we assume to be close to zero. The α's and β's will typically depend on λ, but because we are only concerned with the case $|\lambda| \ll 1$, we fix them at the value they have when $\lambda = 0$.

We assume that each uncoupled oscillator undergoes a supercritical Hopf bifurcation as λ increases through zero. This means that the normal form coefficient, a, for each oscillator is negative. For a system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \qquad x, y \in \mathrm{R}$$

with $f(0,0) = g(0,0) = 0$ and $Df(0,0) = Dg(0,0) = 0$ at a simple Hopf bifurcation

the other components of w are dominated by exponential contraction onto the centre manifold. We know from (3) that

$$w_N = \sum_{j=1}^{N} z_j,$$

so using this and (1) we have

$$\begin{aligned}
\dot{w}_N &= \sum_{j=1}^{N} \dot{z}_j \\
&= \sum_{j=1}^{N} \{(\lambda + i\Omega) z_j + F_2(z_j, \bar{z}_j) + F_3(z_j, \bar{z}_j) + (\epsilon_r + i\epsilon_i)(2z_j - z_{j-1} - z_{j+1})\} \\
&= (\lambda + i\Omega) w_N + \sum_{j=1}^{N} \{F_2(z_j, \bar{z}_j) + F_3(z_j, \bar{z}_j)\} \qquad (4) \\
&= (\lambda + i\Omega) w_N + \frac{F_2(w_N, \bar{w}_N)}{N} + [\dots 2 \dots] + \frac{F_3(w_N, \bar{w}_N)}{N^2} + [\dots 3 \dots]
\end{aligned}$$

where $[\dots 2 \dots]$ represents second order terms in $w_1, \bar{w}_1, \dots, w_{N-1}, \bar{w}_{N-1}$ and $[\dots 3 \dots]$ represents all cubic terms that include at least one of $w_1, \bar{w}_1, \dots, w_{N-1}, \bar{w}_{N-1}$. The second and fourth terms in the last line of (4) were obtained by using the inverse of A,

$$A^{-1}_{jk} = \left(\frac{1}{N}\right) \xi^{(j-1)(N-k)},$$

so that

$$z_j = \sum_{k=1}^{N} A^{-1}_{jk} w_k = \frac{1}{N} \sum_{k=1}^{N} \xi^{(j-1)(N-k)} w_k \qquad (5)$$

and substituting into the expressions for $F_2(z_j, \bar{z}_j)$ and $F_3(z_j, \bar{z}_j)$.

There is a subtle point here regarding the difference between $[\dots 2 \dots]$ and $[\dots 3 \dots]$. We might expect there to be second order terms in (4) of the form $w_N w_k$, $w_N \bar{w}_k$, $\bar{w}_N w_k$ or $\bar{w}_N \bar{w}_k$ for some $k \neq N$, and when we then perform a centre manifold reduction (in which we write w_k and $\bar{w}_k$ as a sum of second–order terms in w_N and $\bar{w}_N$) these terms will be third order in $w_N, \bar{w}_N$. However, it is possible to calculate the coefficients of the second order terms in $w_N w_k$, $w_N \bar{w}_k$, $\bar{w}_N w_k$ and $\bar{w}_N \bar{w}_k$ for any $k \neq N$, and they are all zero. Hence, when we perform the centre manifold reduction all terms in $[\dots 2 \dots]$ are fourth order in $w_N, \bar{w}_N$ and will therefore be ignored. For a similar reason, all terms in $[\ldots 3 \ldots]$ will be of order *at least* 4 (possibly up to 6) and will similarly be ignored. Thus, after performing the centre manifold reduction we can write (4) as

$$\dot{w}_N = (\lambda + i\Omega)w_N + \frac{F_2(w_N, \bar{w}_N)}{N} + \frac{F_3(w_N, \bar{w}_N)}{N^2} + O(|w_N|^4) \tag{6}$$

Comparing this with (1) at $(\epsilon_r, \epsilon_i) = (0, 0)$ we see that to third order they are identical except that the coefficients of the second order terms in (6) have been divided by N and the third order ones by N^2. Going back to the expression for a (equation (2)), we see that the value of a for (6) is equal to that for an uncoupled oscillator divided by N^2. Since only the sign of a is important, we see that in the dynamics restricted to the centre manifold, (6), there is a supercritical Hopf bifurcation as λ increases through 0 when $\epsilon_r < 0$. To see how this manifests itself in the oscillators, we use the inverse of A, (5). Since w_N is small at the onset of oscillation and the w_k for $k \neq N$ are second order in w_N, we can ignore their contribution to the z_j. From (5) we see that $z_j \sim w_N$ for all j, i.e. this branch of orbits manifests itself as the *completely synchronised* state, $z_j = z_k$ for all j, k. The next case we do is $\epsilon_r > 0$, N even.

4 $\epsilon_r > 0$, N even

In this case the first Hopf bifurcation to occur as λ is increased is the simple one for $j = N/2$ at $\lambda = -4\epsilon_r$. The centre manifold is in the $w_{N/2}$ direction and as above, the dynamics in the other directions of w are dominated by exponential contraction onto the centre manifold. From (3) we have

$$w_{N/2} = \sum_{j=1}^{N} \left(A_{\frac{N}{2}j}\right) z_j = \sum_{j=1}^{N} (-1)^{N-j+1} z_j$$

so using (1) we obtain

$$
\begin{aligned}
\dot{w}_{N/2} &= \sum_{j=1}^{N}(-1)^{N-j+1}\dot{z}_j \\
&= \sum_{j=1}^{N}(-1)^{N-j+1}\{(\lambda+i\Omega)z_j + F_2(z_j,\bar{z}_j) + F_3(z_j,\bar{z}_j) + (\epsilon_r + i\epsilon_i)(2z_j - z_{j+1} - z_{j-1})\} \\
&= [\lambda + i\Omega + 4(\epsilon_r + i\epsilon_i)]w_{N/2} + \sum_{j=1}^{N}(-1)^{N-j+1}\{F_2(z_j,\bar{z}_j) + F_3(z_j,\bar{z}_j)\} \\
&= [\lambda + i\Omega + 4(\epsilon_r + i\epsilon_i)]w_{N/2} \\
&+ \frac{1}{N}\left[2\alpha_1 w_{N/2}w_N + \alpha_2(w_{N/2}\bar{w}_N + \bar{w}_{N/2}w_N) + 2\alpha_3\bar{w}_{N/2}\bar{w}_N\right] \\
&+ [\ldots 2 \ldots] + \frac{F_3(w_{N/2},\bar{w}_{N/2})}{N^2} + [\ldots 3 \ldots]
\end{aligned} \tag{7}
$$

where $[\ldots 2 \ldots]$ now represents second order terms in $w_1, \bar{w}_1, \ldots, w_N, \bar{w}_N$ that have *no* factors of $w_{N/2}$ or $\bar{w}_{N/2}$, and $[\ldots 3 \ldots]$ represents cubic terms not composed exclusively of $w_{N/2}$ and $\bar{w}_{N/2}$. After performing the centre manifold reduction, terms in $[\ldots 2 \ldots]$ and $[\ldots 3 \ldots]$ will be of order at least 4 in $w_{N/2}$ and $\bar{w}_{N/2}$ g two [1

Thus (9) becomes

$$[2\sigma_1 w_{N/2} + \sigma_2 \bar{w}_{N/2}] \times [\lambda + i\Omega + 4($$

relevant.

The normal form has three types of nontrivial solution:

1. $u_1 = u_2$, which we associate with the $\mathbf{Z}_2(\kappa)$ orbit,

2. $u_1 = -u_2$, which we associate with the $\mathbf{Z}_2(\kappa, \pi)$ orbit, and

3. either $(u_1, u_2) = (u_1, 0)$ or $(u_1, u_2) = (0, u_2)$, both of which we associate with $\widetilde{\mathbf{Z}}_N$ orbits.

The bifurcation set for the normal form (10) is shown in Figure 1, which is reparametrised version of Figure 3.1 in Ch. XVIII of [6] showing how the bifurcation diagrams for (10) depend on the real parts of B and C. See [6] for more details.

Defining $\xi_- \equiv \xi^{(N-1)/2}$ and $\xi_+ \equiv \xi^{(N+1)/2}$, where $\xi = e^{2\pi i/N}$, we have, using (3)

$$w_q = \sum_{k=1}^{N} \xi_-^{N-k+1} z_k \quad \text{and} \quad w_p = \sum_{k=1}^{N} \xi_+^{N-k+1} z_k$$

Differentiating the first of these with respect to time and using (1) we have

$$\begin{aligned}
\dot{w}_q &= \sum_{k=1}^{N} \xi_-^{N-k+1} \{(\lambda + i\Omega) z_j + F_2(z_j, \bar{z}_j) + F_3(z_j, \bar{z}_j) + (\epsilon_r + i\epsilon_i)(2z_j - z_{j+1} - z_{j+1})\} \\
&= \{\lambda + i\Omega + 2(\epsilon_r + i\epsilon_i)[1 + \cos(\pi/N)]\} w_q + \sum_{k=1}^{N} \xi_-^{N-k+1} \{F_2(z_k, \bar{z}_k) + F_3(z_k, \bar{z}_k)\} \\
&= \{\lambda + i\Omega + 2(\epsilon_r + i\epsilon_i)[1 + \cos(\pi/N)]\} w_q \qquad (11) \\
&+ [2\alpha_1 w_q w_N + \alpha_2(w_q \bar{w}_N + \bar{w}_q w_1) + 2\alpha_3 \bar{w}_q \bar{w}_{N-1}]/N \\
&+ [2\alpha_1 w_p w_1 + \alpha_2(w_p \bar{w}_{N-1} + \bar{w}_p w_N) + 2\alpha_3 \bar{w}_p \bar{w}_N]/N + [\ldots ii \ldots] \\
&+ \frac{\beta_2}{N^2}[w_q^2 \bar{w}_q + 2 w_q w_p \bar{w}_p] + [\ldots iii \ldots]
\end{aligned}$$

where $[\ldots ii \ldots]$ represents second order terms with no factors of $w_q, \bar{w}_q, w_p$ or $\bar{w}_p$, and $[\ldots iii \ldots]$ represents cubic terms excluding those of the form $w_q^2 \bar{w}_q$ and $w_q w_p \bar{w}_p$. When the centre manifold reduction is performed terms in $[\ldots ii \ldots]$ and $[\ldots iii \ldots]$ will be of order at least 4 in $|w_p|$ and $|w_q|$, or if not, can be removed with normal form transformations, and will thus be ignored from now on. We obtain an expression analogous to (11) for $\dot{w}_p$, with w_p and w_q exchanged, as expected from the symmetry of the problem.

The next step is to perform the centre manifold reduction in order to get expressions for w_1, w_N and w_{N-1} in terms of $w_q, \bar{w}_q, w_p$ and $\bar{w}_p$ so that we can substitute them into (11). We write

$$\begin{aligned} w_1 = f(w_q, \bar{w}_q, w_p, \bar{w}_p) &= \gamma_1 w_p^2 + \gamma_2 w_p \bar{w}_p + \gamma_3 w_p w_q + \gamma_4 w_p \bar{w}_q + \gamma_5 \bar{w}_p^2 \\ &+ \gamma_6 \bar{w}_p w_q + \gamma_7 \bar{w}_p \bar{w}_q + \gamma_8 w_q^2 + \gamma_9 w_q \bar{w}_q + \gamma_{10} \bar{w}_q^2 \\ w_N = g(w_q, \bar{w}_q, w_p, \bar{w}_p) &= \theta_1 w_p^2 + \theta_2 w_p \bar{w}_p + \theta_3 w_p w_q + \theta_4 w_p \bar{w}_q + \theta_5 \bar{w}_p^2 \\ &+ \theta_6 \bar{w}_p w_q + \theta_7 \bar{w}_p \bar{w}_q + \theta_8 w_q^2 + \theta_9 w_q \bar{w}_q + \theta_{10} \bar{w}_q^2 \\ w_{N-1} = h(w_q, \bar{w}_q, w_p, \bar{w}_p) &= \nu_1 w_p^2 + \nu_2 w_p \bar{w}_p + \nu_3 w_p w_q + \nu_4 w_p \bar{w}_q + \nu_5 \bar{w}_p^2 \\ &+ \nu_6 \bar{w}_p w_q + \nu_7 \bar{w}_p \bar{w}_q + \nu_8 w_q^2 + \nu_9 w_q \bar{w}_q + \nu_{10} \bar{w}_q^2 \end{aligned} \tag{12}$$

where $\gamma_1, \ldots, \nu_{10} \in \mathrm{C}$ are unknown coefficients. We find them in the usual way — writing equivalent expressions for each of $\dot{w}_1$, $\dot{w}_N$ and $\dot{w}_{N-1}$ and then equating coefficients of like powers of $w_q, \bar{w}_q, w_p$ and $\bar{w}_p$. By substituting the expressions (12) into (11), we can see that after the centre manifold reduction has been performed the coefficient of the term in $w_q^2 \bar{w}_q$ in (11) will be

$$B \equiv \frac{\beta_2}{N^2} + \frac{2\alpha_1\theta_9 + \alpha_2(\bar{\theta}_9 + \gamma_8) + 2\alpha_3\bar{\nu}_{10}}{N} \tag{13}$$

while that of the term in $w_q w_p \bar{w}_p$ will be

$$C \equiv \frac{2\beta_2}{N^2} + \frac{2\alpha_1\theta_2 + \alpha_2\bar{\theta}_2 + 2\alpha_1\gamma_6 + \alpha_2(\bar{\nu}_4 + \theta_3) + 2\alpha_3\bar{\theta}_7}{N} \tag{14}$$

(We have made the correspondence $u_1 = w_q$ and $u_2 = w_p$ for comparison between (11) and (10).) Actually doing the centre manifold reduction, i.e. finding $\gamma_1, \ldots, \nu_{10}$, we see that at the double Hopf bifurcation (i.e. when $\lambda = -2\epsilon_r[1+\cos(\pi/N)]$)

$$\begin{aligned}
\theta_9 &= \frac{i\alpha_2}{\Omega N} + \epsilon_r\left[\frac{2\alpha_2}{\Omega^2 N}(1+\cos(\pi/N))\right] + O(|\epsilon_r,\epsilon_i|^2) \\
\gamma_8 &= -\frac{i\alpha_1}{\Omega N} + \epsilon_r\left[\frac{2\alpha_1}{\Omega^2 N}(\cos(\pi/N)+\cos(2\pi/N))\right] \\
&+ \epsilon_i\left[\frac{2i\alpha_1}{N\Omega^2}(1+2\cos(\pi/N)+\cos(2\pi/N))\right] + O(|\epsilon_r,\epsilon_i|^2) \\
\nu_{10} &= \frac{i\alpha_3}{3\Omega N} + \epsilon_r\left[\frac{2\alpha_3}{9\Omega^2 N}(\cos(\pi/N)+\cos(2\pi/N))\right] \\
&+ \epsilon_i\left[\frac{2i\alpha_3}{9N\Omega^2}(\cos(2\pi/N)-2\cos(\pi/N)-3)\right] + O(|\epsilon_r,\epsilon_i|^2) \\
\theta_2 &= \frac{i\alpha_2}{\Omega N} + \epsilon_r\left[\frac{2\alpha_2}{\Omega^2 N}(1+\cos(\pi/N))\right] + O(|\epsilon_r,\epsilon_i|^2) \\
\gamma_6 &= \frac{i\alpha_2}{\Omega N} + \epsilon_r\left[\frac{2\alpha_2}{\Omega^2 N}(\cos(\pi/N)+\cos(2\pi/N))\right] \\
&+ \epsilon_i\left[\frac{2i\alpha_2}{\Omega^2 N}(\cos(2\pi/N)-1)\right] + O(|\epsilon_r,\epsilon_i|^2) \\
\nu_4 &= \frac{i\alpha_2}{\Omega N} + \epsilon_r\left[\frac{2\alpha_2}{\Omega^2 N}(\cos(\pi/N)+\cos(2\pi/N))\right] \\
&+ \epsilon_i\left[\frac{2i\alpha_2}{N\Omega^2}(\cos(2\pi/N)-1)\right] + O(|\epsilon_r,\epsilon_i|^2) \\
\theta_3 &= -\frac{2i\alpha_1}{\Omega N} + \epsilon_r\left[\frac{4\alpha_1}{\Omega^2 N}(1+\cos(\pi/N))\right] + \epsilon_i\left[\frac{8i\alpha_1}{\Omega^2 N}(1+\cos(\pi/N))\right] + O(|\epsilon_r,\epsilon_i|^2) \\
\theta_7 &= \frac{2i\alpha_3}{3\Omega N} + \epsilon_r\left[\frac{4\alpha_3}{9\Omega^2 N}(1+\cos(\pi/N))\right] + \epsilon_i\left[\frac{-8i\alpha_3}{9N\Omega^2}(1+\cos(\pi/N))\right] + O(|\epsilon_r,\epsilon_i|^2)
\end{aligned}$$

for small $|\epsilon_r|, |\epsilon_i|$. Substituting these expansions into the expressions for B and C (13–14) we obtain

$$\begin{aligned}
Re\{B\} &= \frac{1}{N^2}\left[Re\{\beta_2\} - \frac{Im\{\alpha_1\alpha_2\}}{\Omega}\right] \\
&+ \frac{\epsilon_r}{N^2\Omega^2}\left[2Re\{\alpha_1\alpha_2\}\left(2+3\cos\left(\frac{\pi}{N}\right)+\cos\left(\frac{2\pi}{N}\right)\right)\right. \\
&+ \left. 2|\alpha_2|^2\left(1+\cos\left(\frac{\pi}{N}\right)\right) + \frac{4|\alpha_3|^2}{9}\left(\cos\left(\frac{\pi}{N}\right)+\cos\left(\frac{2\pi}{N}\right)\right)\right] \\
&- \frac{\epsilon_i}{N^2\Omega^2}\left[2Im\{\alpha_1\alpha_2\}\left(1+2\cos\left(\frac{\pi}{N}\right)+\cos\left(\frac{2\pi}{N}\right)\right)\right] + O(|\epsilon_r,\epsilon_i|^2)
\end{aligned} \tag{15}$$

and

$$
\begin{aligned}
Re\{C\} &= \frac{2}{N^2}\left[Re\{\beta_2\} - \frac{Im\{\alpha_1\alpha_2\}}{\Omega}\right] \qquad (16)\\
&+ \frac{\epsilon_r}{N^2\Omega^2}\left[4Re\{\alpha_1\alpha_2\}\left(2 + 3\cos\left(\frac{\pi}{N}\right) + \cos\left(\frac{2\pi}{N}\right)\right)\right.\\
&+ \left. 2|\alpha_2|^2\left(1 + 2\cos\left(\frac{\pi}{N}\right) + \cos\left(\frac{2\pi}{N}\right)\right) + \frac{8|\alpha_3|^2}{9}\left(1 + \cos\left(\frac{\pi}{N}\right)\right)\right]\\
&- \frac{\epsilon_i}{N^2\Omega^2}\left[4Im\{\alpha_1\alpha_2\}\left(1 + 2\cos\left(\frac{\pi}{N}\right) + \cos\left(\frac{2\pi}{N}\right)\right)\right] + O(|\epsilon_r, \epsilon_i|^2)
\end{aligned}
$$

Looking at these expressions when $\epsilon_r = \epsilon_i = 0$, we see that for ϵ_r, ϵ_i small enough, we are in the region $Re\{C\} < Re\{B\} < 0$ of Figure 1 (as

$$Re\{\beta_2\} - \frac{Im\{\alpha_1\alpha_2\}}{\Omega} = a < 0)$$

and thus there is a supercritical Hopf bifurcation to the $\tilde{\mathbf{Z}}_N$ oscillation as λ increases through $-2\epsilon_r[1 + \cos(\pi/N)]$. The $\tilde{\mathbf{Z}}_N$ branch corresponds to two types of oscillation, depending on whether solutions of (11) and its symmetric counterpart are of the form $(w_p, w_q) = (w_p, 0)$ or $(w_p, w_q) = (0, w_q)$. Using (5) we see that if $w_p = 0$ then $z_j \sim \xi_-^{j-1} w_q$, i.e. $z_{j+1} = (\xi_-)z_j$, so the phase difference between neighbouring oscillators is

(Up

because here we have the creation of either a stable $\mathbf{Z}_2(\kappa)$ orbit or a stable $\mathbf{Z}_2(\kappa,\pi)$ orbit in the double Hopf bifurcation. A simple rearrangement of the above expressions for $Re\{B\}$ (15) and $Re\{C\}$ (16) gives

$$\begin{aligned} Re\{C\} &= 2Re\{B\} \\ &+ \frac{\epsilon_r}{N^2\Omega^2}\left[2|\alpha_2|^2\left(\cos\left(\frac{2\pi}{N}\right)-1\right)+\frac{8|\alpha_3|^2}{9}\left(1-\cos\left(\frac{2\pi}{N}\right)\right)\right]+O(|\epsilon_r,\epsilon_i|^2), \end{aligned} \tag{17}$$

so by choosing various parameters correctly, it may be possible to push the normal form for the oscillators from the line $Re\{C\} = 2Re\{B\} < 0$ that we know we are on for $\epsilon_r = \epsilon_i = 0$ across the boundary $Re\{C\} = Re\{B\} < 0$ by increasing ϵ_r, as shown schematically in Figure 2 (compare with Figure 1, which shows the bifurcation diagrams in each sector). (Note that as N increases, the coefficient of the term in ϵ_r in (17) decreases, all other things being equal.) We demonstrate this transition below in an example for $N = 3$, the smallest number of oscillators to have the three different types of orbit created in a double Hopf bifurcation.

We are in the region $\epsilon_r > 0$, and we want to increase $Re\{C\}$, so we set $\alpha_2 = 0$. For simplicity we also set $\epsilon_i, \beta_1, \beta_3$ and β_4 to be zero. The example we use is

$$\dot{z}_j = (\lambda + 1.5i)z_j - 0.7z_j^2 + 2\overline{z}_j^2 - |z_j|^2 z_j + \epsilon(2z_j - z_{j+1} - z_{j-1}) \tag{18}$$

for $j = 1, 2, 3$ and the subscripts are taken mod 3, which corresponds to equation (1) with $\Omega = 1.5, \alpha_1 = -0.7, \alpha_3 = 2, \beta_2 = -1$ and $\epsilon_r = \epsilon$. For this system,

$$Re\{B\} = -\frac{1}{9} + \frac{\epsilon}{9\times 1.5^2}\left[\frac{4\times 2^2}{9}\left(\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{2\pi}{3}\right)\right)\right] + O(\epsilon^2) = -\frac{1}{9} + O(\epsilon^2)$$

so using (17), to first order in ϵ we should get a transition from Hopf bifurcation to a stable $\tilde{\mathbf{Z}}_3$ orbit to Hopf bifurcation to a stable $\mathbf{Z}_2$ orbit of some kind when

$$Re\{C\} - 2Re\{B\} = -Re\{B\}$$

i.e.

$$\frac{\epsilon}{N^2\Omega^2}\left[\frac{8|\alpha_3|^2}{9}\left(1-\cos\left(\frac{2\pi}{N}\right)\right)\right] = \frac{\epsilon}{9\times 1.5^2}\left[\frac{8\times 2^2}{9}\left(1-\cos\left(\frac{2\pi}{3}\right)\right)\right] = \frac{1}{9},$$

i.e. $\epsilon = 27/64 \approx 0.42$. That this transition does occur is demonstrated in Figure 3, where we show the type of orbit that is stable near the double Hopf bifurcation for (18) as a function of ϵ. This clearly shows that for $\epsilon <\sim 0.4$ UaF E ;P

References

[11] A. M. Turing. *The chemical basis of morphogenesis.* Phil. Trans. Roy. Soc. London. **B237**. p.37. 1952.

[12] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer–Verlag Texts in Applied Mathematics. **2**. 1990.

List of Figures

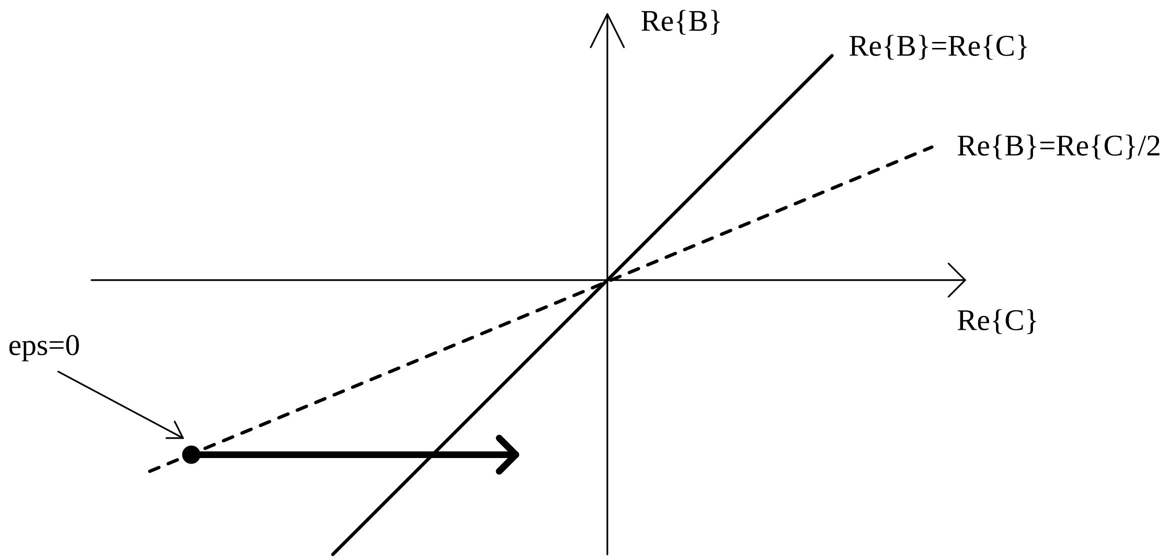

Figure 2: Schematic diagram showing how it might be possible to move from the line $Re\{C\} = 2Re\{B\} < 0$, which we know we are on at $\epsilon_r = \epsilon_i = 0$, across the line $Re\{B\} = Re\{C\} < 0$ by increasing ϵ_r. Compare with Figure 1, which shows bifurcation diagrams for the relevant sectors.

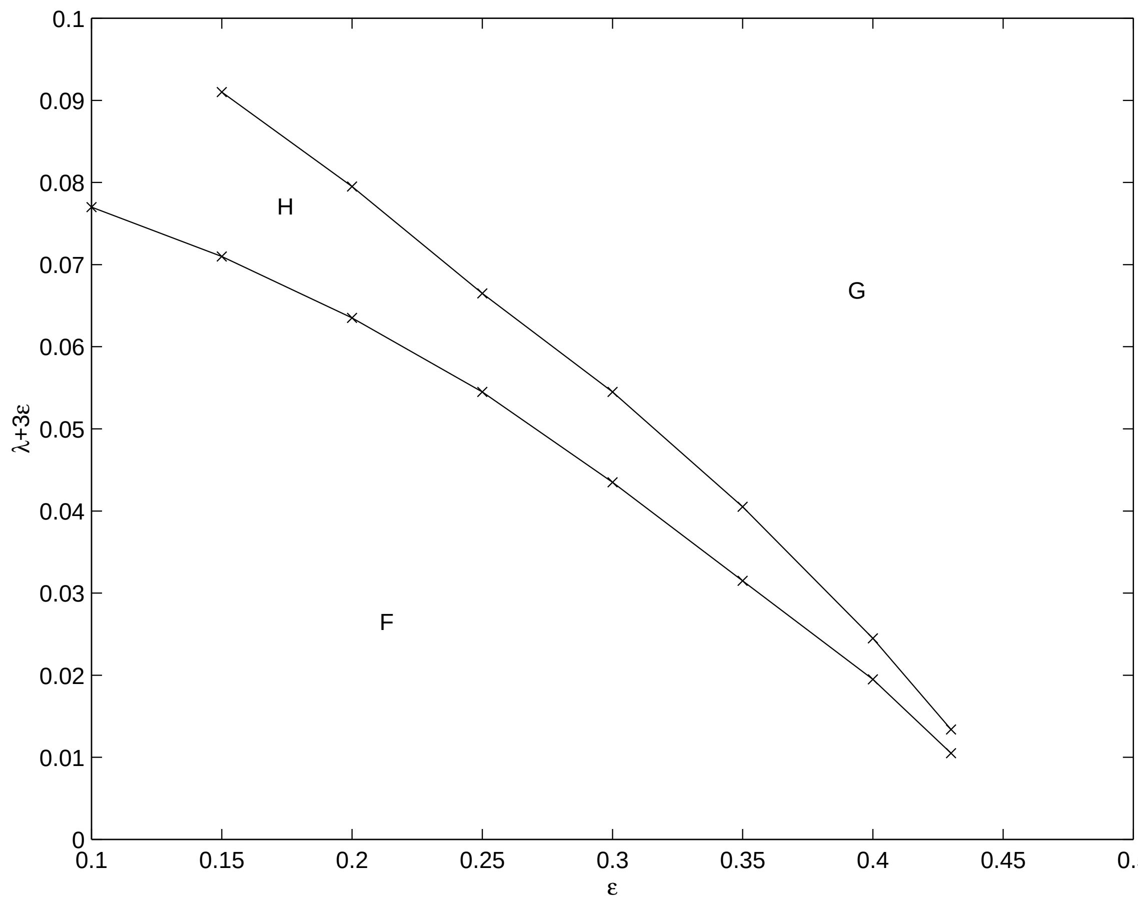

Figure 3: Transition from Hopf bifurcation to a stable $\tilde{\mathbf{Z}}_3$ orbit to Hopf bifurcation to a stable $\mathbf{Z}_2(\kappa, \pi)$ orbit as ϵ is varied in equation (18). In region F the $\tilde{\mathbf{Z}}_3$ orbit is stable, and in region G, the $\mathbf{Z}_2(\kappa, \pi)$ orbit is stable. There is non–periodic behaviour in the wedge H. The vertical coordinate is the distance in λ from the Hopf bifurcation.