List of Figures

[14] J. T. Stuart and R. C. DiPrima. The Eckhaus and Benjamin-Feir resonance mech-

References

[1] P. Ashwin, J. Buescu and I

We have als found t a t this lution, whet er regarded as a s lution of t e periodic problem or the Neumann problem, undergoes a supercritic l blowout bifurc tion. For

the Neumann problem, t is involves breaking only a reflection l symmetry of the lution

while for the periodic problem, this is equiv l ent to a period-increasing bifurc tion. We

believe t is to be t e first observ tion of a blowout bifurc tion from a c a otic lution

ofi a l PDEotaisc cilutiteodnswitre alswidays-ahrmsd inbslte wiithlityespect to perturb tions of period 2π.

with respect to perturbations with period three times that of the solution, we would not expect to see this solution occurring starting with an arbitrary initial condition.

Finally, we consider solutions in $\mathrm{Fix}(\Sigma_4)$ and compute the dominant Lyapunov exponents associated with perturbations corresponding to the two two–dimensional irreducible representations of Σ_4 as a function of μ for $R = 80$, $\nu = 2$. We see for these parameter values that as μ is varied the solution in $\mathrm{Fix}(\Sigma_4)$ changes from periodic or quasiperiodic to chaotic but remains unstable with respect to both types of perturba-

A chaotic solution in $\text{Fix}(\Sigma_1)$ with spatial period π and homogeneous Neumann boundary conditions that is unstable with respect to perturbations of spatial period 2π, i.e. after the blowout bifurcation is shown in Fig. 6. A small perturbation with spatial period 2π is added at $t = 10$ and the symmetry of the solution is quickly lost as expected. The parameter values are $R = 4.2$, $\mu = -4$ and $\nu = 2.8$. Note that only the real part of the solution is shown.

Finally, we consider solutions in $\text{Fix}(\Sigma_2)$ which have spatial period π, are even about the origin and are odd about $\pi/4$. Recall that in this case we are only interested in the Lyapunov exponents of multiplicity two associated with the two-dimensional irreducible representation of Σ_2. In Fig. 7 these dominant Lyapunov exponents are shown. We see that over the parameter range shown, the underlying solution changes from periodic or quasiperiodic to chaotic and back again but is always unstable with respect to perturbations of period 2π. We give an example of such an unstable chaotic solution in Fig. 8. A small perturbation with spatial period 2π is added to the solution at $t = 0.3$. The parameter values are $R = 62$, $\mu = -4$, $\nu = 2$.

For all contour plots, black contour lines correspond to negative values and grey contour lines to positive values.

4.2 Period $2\pi/3$ solutions

In order to investigate the effect of perturbations three times the period of the solution, we computed solutions with period $2\pi/3$, initially with no other symmetries imposed. The dominant Lyapunov exponents associated with the two-dimensional irreducible representation of $\mathbf{Z}_3$ are shown in Fig. 9. This shows a transition to chaos before and after which the solution is unstable with respect to perturbations of period 2π. An example of an unstable chaotic solution corresponding to this parameter range is shown in Fig. 10. Note that at approximately $t = 1.25$, the solution almost has $\mathbf{D}_3$ symmetry but then all symmetry is soon quickly lost after this point.

We next consider solutions in $\text{Fix}(\Sigma_3)$ which have spatial period $2\pi/3$ and are also even about the origin. Again, we only consider the dominant (multiple) Lyapunov exponents associated with the two-dimensional irreducible representation which are shown in Fig. 11 as a function of R for $\mu = -4$ and $\nu = 2$. We see that for this range of parameters the underlying solution is either periodic, quasiperiodic, or chaotic, but is always unstable with respect to perturbations of period 2π. We show an example of such an unstable chaotic solution in Fig. 12 for parameter values $R = 9$, $\mu = -4$ and $\nu = 2$. Again, the black contour lines indicate negative values and the grey contour lines indicate positive values. We note that by rescaling the spatial scale the solution at these parameter values is the same as the solution which was stable with respect to perturbations of period twice that of the solution. Since this solution is not stable occasional “bursts” away from it.

The blowout bifurcation of Fig. 3 seems to be supercritical, as we see bursting behaviour at parameter values close to the bifurcation which is very similar to the on-off intermittency seen in many other examples of blowout bifurcations in low dimensional systems. In Fig. 5 we choose the parameter values $R = 4.2$, $\mu = -4$, $\nu = 2.1667$ and plot the norm of the vector formed from the odd–numbered Fourier coefficients in the spectral representation of the solution as a function of time. The norm is zero if and only if the solution satisfies $A(x,t) = A(x+\pi,t)$. The initial condition was randomly chosen and had spatial period 2π. Thus, for long periods of time, the chaotic motion appears to be even with period π while there are occasional bursts where the period is 2π.

We should also note that the blowout bifurcation does not occur at a particular parameter value but over a range of values. This is typical for a system in which the parameter we vary is *non–normal* [1, 7]. (A non–normal parameter is one for which not only the dynamics normal to the invariant subspace change as we vary the parameter, but also the dynamics restricted to the invariant subspace.)

4 Numerical results

In this section we describe some numerical results relating to the theory presented above. The results are obtained using a pseudo–spectral method as described in [5].

4.1 Period π solutions

Defining $\rho = \beta s_1$, we find that Σ_4 is generated by ρ and s_1 which satisfy

$$s_1^2 = I, \qquad \rho^6 = I, \qquad s_1\rho = \rho^{-1}s_1$$

and so it is isomorphic to $\mathbf{D}_6$. We note that $\beta = \rho s_1$ and $r_\omega = \rho^2$. Now $\mathbf{D}_6$ has four one–dimensional irreducible representations, corresponding to the four combinations of ρ and s_1 being $\pm I$, and two two–dimensional irreducible representations given by

$$\rho = \begin{bmatrix} \cos\omega/2 & -\sin\omega/2 \\ \sin\omega/2 & \cos\omega/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \quad \text{and } s_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad (3.2)$$

and

$$\rho = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \quad \text{and } s_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where $W_1 = \mathrm{Fix}(\mathbf{Z}_3)$. Since the two-dimensional irreducible representation is not absolutely irreducible, there is no further decomposition of the linear operator $g_A(A)$ into two diagonal blocks, as occurred in the previous section with the group $\mathbf{D}_4$. However, it does have a complex structure [15] in that

$$g_A(A)|_{W_2} = \begin{bmatrix} C & -D \\ D & C \end{bmatrix},$$

for some matrices C and D. This implies that if there is a solution $\phi = [u, v]^T$ of the variational equation

$$\dot{\phi} = g_A(A)|_{W_2}\phi, \tag{3.1}$$

then there is also another distinct solution of (3.1) given by $\phi = [-v, u]^T$. Thus the Lyapunov exponents are again of multiplicity two in this case.

The solution with spatial period $2\pi/3$ will be stable to perturbations of period 2π if the (multiple) dominant Lyapunov exponents associated with the isotypic component W_2 are negative.

We now consider solutions which have some reflectional symmetries and have period $2\pi/3$. If solutions are also even about the origin, then the solutions have symmetry group which we call Σ_3 generated by r_ω and s_1 and so is isomorphic to the dihedral group $\mathbf{D}_3$. This group has two one–dimensional irreducible representations $r_\omega = I, s_1 = I$ and $r_\omega = I, s_1 = -I$, and one two–dimensional representation

$$r_\omega = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \quad \text{and } s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In a similar way to the $\mathbf{D}_4$ case above, r_ω acts as the identity for the one–dimensional irreducible representations and so perturbations in the corresponding isotypic components have the same period as the solution. Thus, only the two-dimensional irreducible representation is of interest and since it is also absolutely irreducible, the Lyapunov exponents associated with the corresponding isotypic component will have multiplicity two. Again numerically it is sufficient to consider only perturbations for which $s_1 = I$.

The theory is again similar for solutions which have spatial period $2\pi/3$ and are odd about the origin.

Finally, we consider solutions which are even about the origin, odd about $\pi/6$ and have period $2\pi/3$. It is helpful to define

$$\beta A(x,t) := s_2 s_1 r_{\pi/3} A(x,t) = -A(\pi/3 - x, t),$$

since functions fixed by β are odd about $\pi/6$. The symmetry group Σ_4 of these solutions thus includes s_1 (even about the origin), β (odd about $\pi/6$) and r_ω (period $2\pi/3$).

represent different combinations of reflectional symmetries being broken which preserve the period, which we considered in [

isotypic component. Thus,

$A(x, t)$

The Fourier decompositions of solutions in W_1 and W_2 are

$$A(x,t) \in \ /$$

corresponding to, respectively, a rotation of the complex amplitude, space translation, time translation and a spatial reflection. We note that a special case of the rotation occurs when $\theta = \pi$ and this gives another symmetry of order two. As in [5], we define

$$\pi A(x,t) := s_2 A(x,t) = -A(x,t).$$

As we are interested in spatial period increasing bifurcations, we consider the CGL equation on the spatial domain $[0, 2\pi]$ together with periodic boundary conditions but we consider solutions with period $2\pi/n$ for some integer $n > 1$. Thus, perturbations with period 2π, the domain length, represent an increase in the period by a factor of n. Clearly such solutions are invariant under a translation of their period $2\pi/n$ and so are contained in $\mathrm{Fix}(\mathbf{Z}_n)$ where $\mathbf{Z}_n$ is the cyclic group of order n generated by $r_{2\pi/n}$. We will also consider solutions which have in addition some reflectional symmetries.

We noted in [5] that the CGL equation usually has three zero Lyapunov exponents. However, these are all associated with isotypic components which do not involve an increase in the period and so are not relevant in this context.

3 Period Increasing Bifurcations

We consider solutions with period $2\pi/n$ for particular values of n. We will concentrate on the values of $n = 2$ and $n = 3$ since then the generalisation to higher values of n will be obvious.

3.1 Spatial period doubling ($n = 2$)

When $n = 2$, the solutions that we are interested in have spatial period π and so are fixed by the action of r_π. If the solutions have no other symmetries then they are contained in $\mathrm{Fix}(\mathbf{Z}_2)$. The corresponding isotypic decomposition is simply

$$X = W_1 \oplus W_2$$

where

$$\begin{aligned} W_1 &= \{A \in X : r_\pi A = A\} = \mathrm{Fix}(\mathbf{Z}_2) \\ W_2 &= \{A \in X : r_\pi A = -A\}. \end{aligned}$$

that g satisfies the equivariance condition

$$\gamma g(A) = g(\gamma A) \quad \text{for all } \gamma \in \Gamma, \tag{2.2}$$

where Γ is a compact Lie group. For any subgroup Σ of Γ, we define the fixed point space

$$\text{Fix}(\Sigma) = \{A \in X : \sigma A = A \text{ for all } \sigma \in \Sigma\}$$

and it is easily verified that

$$g : \text{Fix}(\Sigma) \to \text{Fix}(\Sigma)$$

for all subgroups Σ of Γ so that the fixed point spaces are invariant under the flow of the nonlinear equation (2.1).

For each subgroup Σ of Γ, there is a unique Σ-isotypic decomposition of the space X given by

$$X = \sum_k \oplus W_k,$$

where each isotypic component W_k is the sum of irreducible subspaces which are associated with one of the irreducible representations of Σ. If there is a solution $A(t) \in \text{Fix}(\Sigma)$ of (2.1), then the Σ-isotypic components are invariant under the linearisation of g about $A(t)$, i.e.

$$g_A(A(t)) : W_k \to W_k$$

and so there is a block diagonal structure to the linear operator $g_A(A(t))$. Since this linear operator is involved in the variational equation which is used to compute Lyapunov exponents, we can associate Lyapunov exponents with a particular isotypic component. There are two important consequences of this decomposition which are as follows:

1. the Lyapunov exponents can be calculated for perturbations in each of the isotypic components independently;

2. the motion in $\text{Fix}(\Sigma)$ will be stable if the dominant Lyapunov exponent associated with each of the isotypic components other than the trivial one (which is $\text{Fix}(\Sigma)$) are negative.

We apply these ideas to the CGL equation (1.1) which has a number of symmetries given by

$$\begin{aligned}
\theta A(x,t) &= e^{i\theta} A(x,t), & \theta \in [0, 2\pi) \\
r_\alpha A(x,t) &= A(x+\alpha, t), & \alpha \in [0, 2\pi) \\
\tau_\beta A(x,t) &= A(x, t+\beta), & \beta \in \mathbb{R} \\
s_1 A(x,t) &= A(-x, t), &
\end{aligned}$$

odd perturbations.

In this paper we continue the investigation, but consider solutions that have a spatial period L and investigate their stability with respect to perturbations that have spatial period kL for some integer $k > 1$, i.e. *spatial period increasing* perturbations. These are often referred to as *side-band* perturbations. (Note that this is quite different from the ideas of period–doubling or multiplying that have gained much attention in the past 20 years

1 Introduction

The formation of patterns in the solutions of partial differential equations which model many physical systems has been the subject of much interest over many decades. Associated with this are ideas of *self organisation* in which particular patterns are chosen by a particular system and this is determined by the stability of different patterns since only stable solutions will be seen in practice. Mathematically speaking, solutions of an equation are found in a particular function space. The question of stability can be a delicate one since it is often necessary to consider the effects of small perturbations on the solution which are not in the same space as the solution. A simple example is when a solution has certain symmetry properties but such a solution may be unstable to perturbations which break the symmetry of the solution.

Studies in pattern formation are usually concerned with either steady state or time periodic solutions of PDE's and patterns are often associated with symmetries of the solutions [11]. However, we consider patterns that occur in spatio-temporally chaotic solutions of PDE's, which are defined in terms of their symmo ll pefloncerned wi ermo l stem

Symmetry and Chaos in the Complex Ginzburg–Landau Equation. II: Translational symmetries

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Abstract

The complex Ginzburg–Landau (CGL) equation on a 1–dimensional domain with periodic boundary conditions has a number of different symmetries, and solutions of the CGL may or may not be fixed by the action of these symmetries. In this paper we investigate the stability of chaotic solutions that are spatially periodic but have a period that is some fraction of the domain length, L, with respect to perturbations that have a spatial wavelength equal to the domain length. We do this by considering the isotypic decomposition of the space of solutions and finding the dominant Lyapunov exponent associated with each isotypic component.

We find a region of parameter space in which there exist chaotic solutions with spatial period $L/2$ and homogeneous Neumann boundary conditions that are stable with respect to perturbations of period L. On varying the parameters in the at9 d g t c cer
pe ri3 fo I w